MATH 207(c)–Analysis IV

1. Show that, for any $a, \omega \in \mathbb{R}$, the Laplace transform of the functions $f(t) = e^{at}$, $g(t) = \cos(\omega t)$ and $h(t) = \sin(\omega t)$ are given by the expressions

$$\mathcal{L}[f](z) = \frac{1}{z-a}, \quad \mathcal{L}[g](z) = \frac{z}{z^2+\omega^2} \quad \text{and} \quad \mathcal{L}[h](z) = \frac{\omega}{z^2+\omega^2}$$

Can you find an abscissa of convergence? What is the optimal (i.e. minimum) abscissa of convergence in each case?

2. Using the properties of the Laplace transform that we saw in class, show that the indicated γ_0 is an abscissa of convergence and compute the Laplace transforms of the following functions:

(a) $f(t) = t \cdot e^{at}$ (for $a \in \mathbb{R}$), $\gamma_0 = a$.

(b) $f(t) = e^{z_0 t}$ (for $z_0 \in \mathbb{C}$), $\gamma_0 = \mathrm{Re}(z_0)$.

(c) $f(t) = t^2 \cos(\omega t)$ (for $\omega \in \mathbb{R}$), $\gamma_0 = 0$.

(d) $f(t) = e^{-2t}\big(\cos(3t) - \sin(5t)\big)$, $\gamma_0 = -2$.

(e) $f(t) = te^{at}\cos(\omega t)$ (for $a \in \mathbb{R}, \omega > 0$), $\gamma_0 = a$.

3. Let $f, g : [0, +\infty) \to \mathbb{R}$ be the functions

$$f(t) = t \quad \text{and} \quad g(t) = e^{-t}.$$

Compute the convolution $f * g$ as well as its Laplace transform.

4. Let us consider the following boundary value problem: For a given continuous function $f : [0,1] \to \mathbb{R}$,

$$\begin{cases} y''(t) = f(t) & \text{for } t \in [0,1], \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

We will solve the above using the Laplace transform.

(a) Let $Y(z)$ and $F(z)$ be the Laplace transforms of y and f, respectively (how should they be defined, given that, originally, y, f are only defined on $[0,1]$?). Show that we must have

$$Y(z) = \frac{F(z)}{z^2} + \frac{y'(0)}{z^2}.$$

(b) Deduce that

$$y(t) = \int_0^t f(s)(t-s)\,ds + ty'(0).$$

(c) Show that

$$y(t) = \int_0^t f(s)(t-s)\,ds - t\int_0^1 f(s)(1-s)\,ds.$$

5. Consider the following system of differential equations:

$$\begin{cases} x'(t) = 2x(t) - 3y(t), \quad t > 0, \\ y'(t) = -2x(t) + y(t), \quad t > 0, \\ x(0) = 8, \quad y(0) = 3. \end{cases}$$

We will denote the Laplace transforms of x, y by $X(z) = \mathcal{L}[x](z)$ and $Y(z) = \mathcal{L}[y](z)$, respectively.

(a) Applying the Laplace transform, show that X and Y satisfy a linear system of the form

$$\begin{aligned} (z+a)X(z) + bY(z) &= 8, \\ cX(z) + (z+d)Y(z) &= 3, \end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$. Determine the values of a, b, c, d and solve for $X(z)$ and $Y(z)$.

(b) Conclude by finding an expression for $x(t)$ and $y(t)$.